ISSN 1464-8997 (on line) 1464-8989 (printed)

Geometry & Topology Monographs
Volume 1: The Epstein birthday schrift
Pages 479–492

The boundary of the deformation space of the fundamental group of some hyperbolic 3–manifolds fibering over the circle

Leonid Potyagailo

Abstract By using Thurston's bending construction we obtain a sequence of faithful discrete representations ρ_n of the fundamental group of a closed hyperbolic 3–manifold fibering over the circle into the isometry group $Iso\,\mathbf{H}^4$ of the hyperbolic space $\mathbf{H}^4$. The algebraic limit of ρ_n contains a finitely generated subgroup F whose 3–dimensional quotient $\Omega(F)/F$ has infinitely generated fundamental group, where $\Omega(F)$ is the discontinuity domain of F acting on the sphere at infinity $S^3_\infty = \partial\mathbf{H}^4$. Moreover F is isomorphic to the fundamental group of a closed surface and contains infinitely many conjugacy classes of maximal parabolic subgroups.

AMS Classification 57M10, 30F40, 20H10; 57S30, 57M05, 30F10, 30F35

Keywords Discrete (Kleinian) subgroups, deformation spaces, hyperbolic 4–manifolds, conformally flat 3–manifolds, surface bundles over the circle

1 Introduction and statement of results

By a Kleinian (discontinuous) group G we mean a subgroup of the group $\mathrm{Conf}(\mathbf{S}^n) \cong SO_+(1, n+1)$ of conformal transformations of $\overline{R}^n = S^n = R^n \cup \{\infty\}$ which acts discontinuously on a non-empty set $\Omega(G) \subset S^n$ called its domain of discontinuity. It may be connected or not; we will say that G is a function group if there is a connected component $\Omega_G \subset \Omega(G)$ that is invariant under the action of the whole group: $G\Omega_G = \Omega_G$. The quotient spaces $M_G = \Omega_G/G$ and $M(G) = \Omega(G)/G$ are n–manifolds in the case in which G is torsion-free. The complement $\Lambda(G) = (S^n \backslash \Omega(G)) \subset \partial\mathbf{H}^{n+1}$ is called the limit set of G.

A finitely generated Kleinian group G is called geometrically finite if for some $\varepsilon > 0$ there exists an ε–neighbourhood of H_G/G in $\mathbf{H}^{n+1}/G$ which is of finite hyperbolic volume. Here $H_G \subset \mathbf{H}^{n+1}$ is the convex hull of $\Lambda(G)$.

Let us consider for $n = 3$ a hyperbolic 3–manifold $M = H^3/\Gamma$ $(\Gamma \subset PSL_2\mathbf{C})$ fibering over the circle S^1 with fiber a closed surface σ. The notation is $M = \sigma\tilde{\times}S^1$. A representation $\rho\colon \pi_1(M) \to \mathrm{Conf}(\mathbf{S}^3)$ is called admissible if the following conditions are satisfied.

(1) $\rho\colon \Gamma \to \mathrm{Conf}(\mathbf{S}^3)$ is faithful and $\rho(\Gamma) = \Gamma_0$ is Kleinian.

(2) ρ preserves the type of each element, ie $\rho(\gamma)$ is loxodromic for all $\gamma \in \Gamma$.

(3) ρ is induced by a homeomorphism $f_\rho\colon \Omega(\Gamma) \to \Omega(\Gamma_0)$, namely $f_\rho \gamma f_\rho^{-1} = \rho(\gamma)$, $\gamma \in \Gamma$.

The set of all admissible representations modulo conjugation in $\mathrm{Conf}(\mathbf{S}^3)$ is called the deformation space $\mathrm{Def}(\Gamma)$ of the group Γ.

The set $\mathrm{Def}(\Gamma)$ inherits the topology of convergence on generators of Γ on compact subsets in $\mathbf{S}^3$ because $\mathrm{Def}(\Gamma) \subset \left(\mathrm{Conf}(\mathbf{S}^3)\right)^k/\sim$, $k \in \mathbf{N}$ ($\sim$ is conjugation in $\mathrm{Conf}(\mathbf{S}^3)$). As $\mathrm{Def}(\Gamma)$ is a bounded domain [13] two questions have arisen. The first is to describe the cases when $\mathrm{Def}(\Gamma)$ is non-trivial and the second is to study the boundary $\partial\,\mathrm{Def}(\Gamma)$, as was done for the classical Teichmüller space [2], [10]. The answer to the first question is still unknown even in the case when M is Haken. We will consider the case when M contains many totally geodesic surfaces. Each of them produces a curve in $\mathrm{Def}(\Gamma)$ by Thurston's "bending" construction [19]. Our main interest is in groups which appear on the boundary $\partial\,\mathrm{Def}(\Gamma)$. These are higher dimensional analogs of B–groups which arise as the limits of sequences of quasifuchsian groups in classical Teichmüller space.

One of the most fundamental questions is to describe the topological type of the orbifold $M(\Gamma) = \Omega(\Gamma)/\Gamma$ (a manifold in the case when Γ is torsion-free), in particular, when Γ is a function group it is important to know when the fundamental group $\pi_1(M_G = \Omega_\Gamma/\Gamma)$ turns out to be finitely generated, or even more generally when it has finite homotopy type.

In dimension 2 the famous theorem of Ahlfors [1] says that a finitely generated non-elementary Kleinian group $G \subset \mathrm{Conf}(\mathbf{R}^2)$ has a factor-space $\Omega(G)/G$ consisting of a finite number of Riemann surfaces $S_1, \ldots, S_n$ each having a finite hyperbolic area.

We discovered in [7] that the weakest topological version of Ahlfors' theorem does not hold starting already with dimension 3. Namely we constructed a finitely generated function group $F \subset \mathrm{Conf}(\mathbf{S}^3)$ such that the group $\pi_1(\Omega_F/F)$ is not finitely generated. Afterwards it was pointed out in [15] that this group is in fact not finitely presented.

It has also been shown that there exists a finitely generated Kleinian group with infinitely many conjugacy classes of parabolics [6].

In [14] we constructed a finitely generated group F_1 such that $\pi_1(\Omega_{F_1}/F_1)$ is not finitely generated and having infinitely many non-conjugate elliptic elements; moreover F_1 appears as an infinitely presented subgroup of a geometrically finite Kleinian group in $\mathbf{H}^4$ without parabolic elements. On the other hand, it was shown in [4] that a finitely generated but infinitely presented group can also appear as a subgroup of a cocompact group in $SO(1,4)$.

Theorem 1 *Let $\Gamma = \pi_1(M)$ be the fundamental group of a hyperbolic 3–manifold M fibering over the circle with fiber a closed surface σ. Suppose that Γ is commensurable with the reflection group R determined by the faces of a right-angular polyhedron $D \subset \mathbf{H}^3$. Then there exists a finite-index subgroup $L \subset \Gamma$ and a path $\beta_t\colon [0,1[\mapsto \mathrm{Def}(\Gamma)$ such that β_t converges to a faithful representation $\beta_1 \in \partial\mathrm{Def}(\Gamma)$ (as $t \to 1$) and the following hold:*

(1) *$\beta_1(F_L)$ contains infinitely many conjugacy classes of maximal parabolic subgroups,*

(2) *$\pi_1(\Omega_{\beta_1(F_L)})/\beta_1(F_L)$ is infinitely generated,*

where $F_L = L \cap \pi_1\sigma$ is isomorphic to the fundamental group of a closed hyperbolic surface which finitely covers σ and $\beta_1(F_L)$ acts discontinuously on an invariant component $\Omega_{\beta_1(F_L)} \subset \mathbf{S}^3$.

Remark Groups satisfying all the conditions of Theorem 1 do exist. An example of Thurston, of the reflection group in the faces of the right-angular dodecahedron, which is commensurable with a group of a closed surface bundle, is given in [18].

Acknowledgement This paper was prepared several years ago while the author had a Humboldt Fellowship at the Rühr-Universität in Bochum. The author is deeply grateful to Heiner Zieschang and to the Humboldt Foundation for this opportunity. I would also like to thank Nicolaas Kuiper (who died recently) for reading a preliminary version of the manuscript and to express my gratitude to the referee for many useful remarks and corrections.

2 Outline of the proof

Before giving a formal proof of the Theorem let us describe it informally.

Our construction is inspired essentially by papers [6], [8] and [14]. In the first two a free Kleinian group of finite rank satisfying the conclusion (2) was produced, whereas now we give an example of a closed surface group with this property. Our present construction is essentially easier than that of [14]. Also, we produce a curve in the deformation space whose limit point is the group in question.

Step 1 We start with an uniform lattice $\Gamma \subset PSL_2\mathbf{C}$ commensurable with the reflection group R whose limit set is the Euclidean 2–sphere ∂B_1 – the boundary of the ball $B_1 \subset \mathbf{S^3}$. There exists a Fuchsian subgroup $H_2 \subset \Gamma$ leaving invariant a vertical plane π whose intersection with B_1 is a round circle, its limit set $\Lambda(H_2)$ (see figure 1). The group H_2 also leaves invariant a geodesic plane $w_2 \subset B_1$. Consider the action of the group Γ in the outside ball $B_1^* = \mathbf{S^3} \backslash B_1$. For some finite-index subgroup Γ_1 of Γ we construct a new group G_1 obtained by Maskit's Combination theorem from Γ_1 and $\tau_\pi \Gamma_1 \tau_\pi$ combined along the common subgroup $H_2 = \mathrm{Stab}\, w_2$, where τ_π is the reflection in π. The new group G_1 is still isomorphic to some subgroup $G^* \subset R$ of finite index essentially because the same construction can be done inside B_1 by reflecting the picture along the geodesic plane w_2. Thus G_1 belongs to the deformation space $\mathrm{Def}(G_1^*)$. One can obtain a fundamental domain $R(G_1) \subset B_1^*$ of G_1 which is situated in a small neighbourhood of the spheres ∂B_1 and $\tau_\pi(\partial B_1)$.

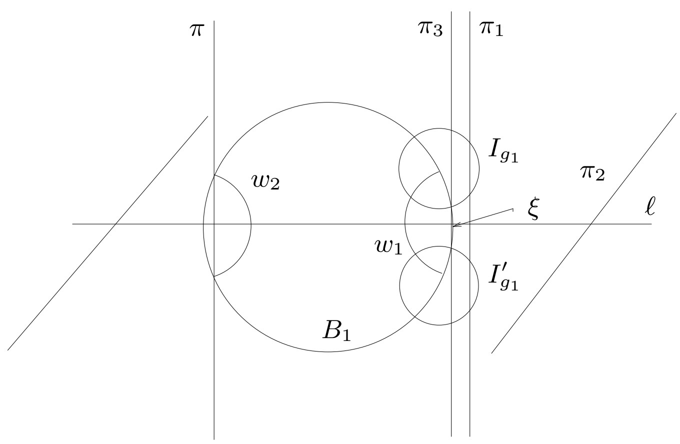

Figure 1

Step 2 There is another geodesic plane $w_1 \subset B_1$ disjoint from w_2 whose stabilizer in Γ_1 is H_1 (see figure 2). Denote by B_2 the ball $\tau_\pi(B_1)$. Take a sphere $\Sigma \subset B_1^*$ passing through the circle $w_3 \cap B_2$ – the limit set of the group $\tau_\pi H_1 \tau_\pi$ – and tangent to the isometric spheres of some element $g_1 \in \Gamma_1$, where H_1 is a subgroup of Γ_1 stabilizing w_1. We now construct a family of Euclidean spheres Σ_t $(0 \le t \le 1,\ \Sigma_1 = \Sigma)$ and corresponding groups $\mathcal{G}_t$ obtained as before from G_1 and $\tau_{\Sigma_t} G_1 \tau_{\Sigma_t}$ by using the combination method along common closed surface subgroups. We prove then that there is a path $\beta_t\colon t \in [0,1[\ \mapsto \beta \in \mathrm{Def}(L')$ such that $\beta_0 = L'$, $\beta_t = \mathcal{G}_t$ where L' is some finite-index subgroup of R. One can equally say that β_t is obtained by using Thurston's bending deformation. The main point is now to prove that the limit

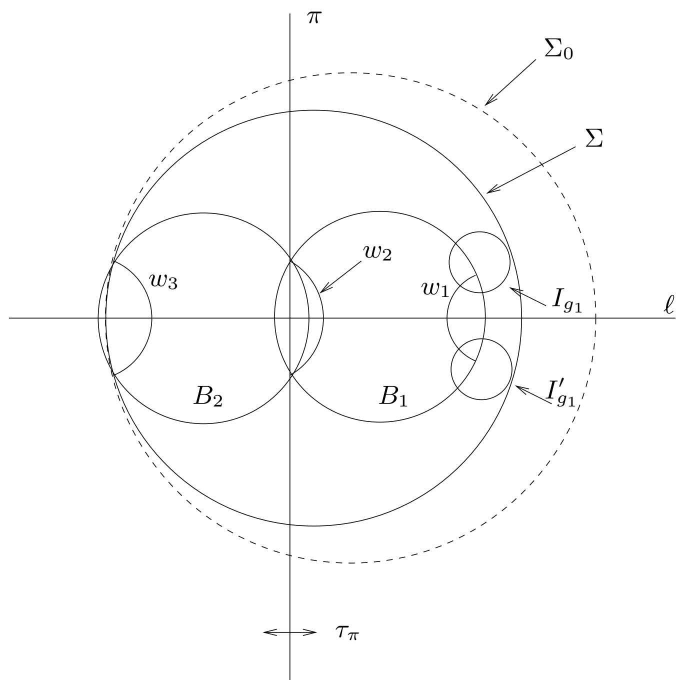

Figure 2

group $\mathcal{G}_1 = \lim_{t\to 1} \beta_t(L')$ is discontinuous and has a fundamental domain obtained from the part of $R(G_1)$ by doubling along the sphere Σ. The group $\mathcal{G}_1$ is also isomorphic to L' and so contains a fundamental group $\mathcal{N}$ of a closed surface bundle over the circle which is isomorphic to the group $L = \Gamma \cap L'$. Let $\mathcal{F}$ be the fundamental group of the fiber given by $\beta_1(F_L = F \cap L)$. Since two isometric spheres of the element $g_1 \in \Gamma_1$ are tangent to Σ, we get a new accidental parabolic element $g = g_1 \cdot g_2$, $g_2 = \tau_\Sigma g_1 \tau_\Sigma$ in the group $\mathcal{G}_1$. By a choice of g_1 made from the very beginning we assure that $g \in \mathcal{F}$, so we have a pseudo-Anosov action of some element $t \in \mathcal{N} \setminus \mathcal{F}$ such that the orbit $t^n \cdot g \cdot t^{-n}$ $(n \in \mathbf{Z})$ gives us infinitely many conjugacy classes of maximal parabolic subgroups of $\mathcal{F}$. Now Scott's compact core theorem implies that $\pi_1(\Omega_{\mathcal{F}})/\mathcal{F}$ is not finitely generated. *End of outline*

3 Preliminaries

We will consider the Poincaré model of hyperbolic space $\mathbf{H}^3$ in the unit ball B_1 equipped with the hyperbolic metric ρ. By a right-anguled polyhedron $D \subset \mathbf{H}^3$ we mean a polyhedron all of whose dihedral angles are $\pi/2$.

Consider the tesselation of $\mathbf{H^3}$ by images of D under the reflection group R from Theorem 1. Denote by $W \subset \mathbf{H}^3$ the collection of geodesic planes w such that there exists $r \in R$, for which $r(w) \cap \partial D$ is a face of D.

It is easy to see that if σ_1 and σ_2 are two faces of D with $\sigma_1 \cap \sigma_2 = \emptyset$, then also the geodesic planes $\tilde{\sigma}_1 \supset \sigma_1$ and $\tilde{\sigma}_2 \supset \sigma_2$ have no point in common. One can easily show that the distance between σ_1 and σ_2, as well as that of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, is realized by a common perpendicular ℓ for which $\ell \cap \mathrm{int} D \neq \emptyset$.

Let $\Gamma_0 = R \cap \Gamma$ which is a subgroup of a finite index in both groups R and Γ. By passing to a subgroup of a finite index and preserving notation, we may assume that Γ_0 is a normal subgroup in R, $|R : \Gamma_0| < \infty$. For a plane $w \in W$ we write $H_w = \mathrm{Stab}(w, \Gamma_0) = \{g \in \Gamma_0,\ gw = w\}$. It is not hard to see that H_w is a Fuchsian group of the first kind commensurable with the reflection group determined by the edges of some face of the polyhedron $r(D_1),\ r \in R$.

Let us now fix two disjoint planes w_1 and w_2 from W containing opposite faces of D and let ℓ be their common perpendicular; up to conjugation in Isom $\mathbf{H}^3$ we can assume that ℓ is a Euclidean diameter of B_1. Denote $B_1^* = \mathbf{S}^3 \backslash cl(B_1)$ as well (where $cl(\cdot)$ is the closure of a set). We have the following:

Lemma 1 *For every horosphere π_3 in B_1^* centered at the point $\xi \in \ell \cap \partial B_1$ (see figure 1) there exists $\varepsilon_0 > 0$ such that for every ε–close sphere $\pi_1 \subset B_1^*$ to π_3 ($\varepsilon < \varepsilon_0$) orthogonal to the plane π_2 there exists a geodesic plane w and an element $g_1 \in [H_w, H_w]$ (commutator subgroup) such that:*

$$
\begin{aligned}
&I_{g_1} \cap \pi_1 \neq \emptyset \quad \text{and} \quad g_1(I_{g_1} \cap \pi_1) = I'_{g_1} \cap \pi_1, \\
&\text{where} \quad I_{g_1},\ I'_{g_1} = I_{g_1^{-1}} \quad \textit{are isometric spheres of } g_1.
\end{aligned}
\tag{1}
$$

Proof Up to further conjugation in Isom B_1 preserving ℓ we may assume that π_3 is the vertical plane tangent to ∂B_1 at $\xi \in \ell \cap \partial B_1$. Take $w = w_1$ and let $g_1 \in [H_{w_1}, H_{w_1}]$ be any primitive element corresponding to a simple dividing loop on the surface w_1/H_{w_1}.

Suppose first that $I_{g_1} \cap \pi_3 = \emptyset$. In this case we proceed as follows. Put $\chi = \tau_{w_1} \circ \tau_{w_2} \in R$, where τ_{w_i} denotes the reflection in plane w_i $(i = 1, 2)$. Then χ is a hyperbolic element whose invariant axis is ℓ. Consider the sequence of planes $\chi^n(w_1)$. We claim that, for some n, $\chi^n(I_{g_1}) \cap \pi_3 \neq \emptyset$. In fact this follows directly from the fact that the fixed point ξ of the hyperbolic element χ is a conical limit point of Γ_0, and so the approximating sequence $\chi^n(I_{g_1})$ should intersect a fixed horosphere (or equivalently by sending ξ to the infinity and passing to the half-space model one can see that χ becomes now a dilation $z \mapsto \lambda z$ $(\lambda > 0)$ which implies that the translations of the image of I_{g_1} by

powers of the dilation will intersect a fixed horosphere at infinity). Since Γ_0 is normal in R it now follows that $\chi^n g_1 \chi^{-n} \in [H_{\chi^n(w_1)}, H_{\chi^n(w_1)}] \subset \Gamma_0$ and $\chi^n(I_{g_1}) = I_{\chi^n g_1 \chi^{-n}}$. The latter is true since χ preserves each Euclidean plane passing through $B_1 \cap \ell$ and, hence $(\chi^n g_1 \chi^{-n})|_{\chi^n(I_{g_1})}$ is an Euclidean isometry. So up to replacing w_1 by $\chi^n(w_1)$ and g_1 by $\chi^n g_1 \chi^{-n}$ if needed, we may assume that $I_{g_1} \cap \pi_3 \neq \emptyset$. The same conclusion is then obviously true for a plane $\pi_1 \subset B_1^*$ sufficiently close to π_3.

For $\ell_1 = I_{g_1} \cap \pi_1$ we now claim that $g_1(\ell_1) = \ell_2 = I'_{g_1} \cap \pi_1$. Indeed, $g_1 = \tau_{\pi_2} \cdot \tau_{I_{g_1}}$ where π_2 is orthogonal to π_1 and contains ℓ (figure 1). Evidently

$$g_1(\ell_1) = \tau_{\pi_2}(I_{g_1} \cap \pi_1) = \tau_{\pi_2}(I_{g_1}) \cap \pi_1 = I'_{g_1} \cap \pi_1 \tag{2}$$

since $\tau_{\pi_2}(\pi_1) = \pi_1$. The lemma is proved. $\square$

So we can suppose that $w_1 \in W$ is chosen satisfying all the conclusions of Lemma 1. Let $w_2 \in W$ be a geodesic plane disjoint from w_1 and let ℓ be their common perpendicular passing through the origin of B_1. Now consider the Euclidean plane π orthogonal to ℓ (figure 2) such that

$$\pi \cap \partial B_1 = \pi \cap w_2 \ .$$

It is not hard to see that $\mathrm{Stab}(\pi, \Gamma) = \mathrm{Stab}(w_2, \Gamma) = H_{w_2}$. Reflecting our picture in the plane π we get

$$B_2 = \tau_\pi(B_1) \ , \quad w_3 = \tau_\pi(w_2) \quad \text{and}$$
$$H_{w_3} = \tau_\pi H_{w_1} \tau_\pi \ .$$

By Lemma 1 we can now find a Euclidean sphere Σ centered on ℓ which goes through the circle $w_3 \cap \partial B_2$ and is tangent to I_{g_1} (figure 2). Moreover, by Lemma 1, Σ is tangent also to I'_{g_1}.

Denote $\Sigma' = \tau_\pi^{-1}(\Sigma)$.

Lemma 2 *There exists a subgroup $\Gamma_1 \subset \Gamma_0$ of finite index such that the following conditions hold:*

(a) *The boundary of the isometric fundamental domain $\mathcal{P}(\Gamma_1) \subset B_1^*$ lies in a regular ε–neighbourhood of ∂B_1^* $\big(B_1^* = \mathbf{S}^3 \backslash cl(B_1),\ \varepsilon > 0\big)$.*

(b) $\Sigma \cap I_\gamma = \emptyset \ , \quad \gamma \in \Gamma_1 \backslash \{g_1, g_1^{-1}\}$.

(c) *For subgroups $H_1 = \Gamma_1 \cap H_{w_1}, H_2 = \Gamma_1 \cap H_{w_2}$ there exists another fundamental domain $R(\Gamma_1) \subset B_1^*$ of Γ_1 such that*

$$R(\Gamma_1) \cap (\pi \cup \Sigma') = \mathcal{P}(H) \cap (\pi \cup \Sigma'),$$

where $\mathcal{P}(H)$ is an isometric fundamental domain for the group $H = \langle H_1, H_2 \rangle$.

(d) $g_1 \in \Gamma_1 \cap [H_1, H_1]$.

Proof This Lemma can be obtained by repeating the arguments of [14, Main Lemma]. We just sketch these considerations. First, we choose a subgroup $\tilde{\Gamma} \subset \Gamma_0$ of a finite index satisfying conditions (a) and (b) such that $g_1 \in \tilde{\Gamma}$ by using the property of separability of infinite cyclic subgroups in Γ_0 [9].

To obtain (c) we will find Γ_1 by using Scott's $LERF$–property of the group Γ_0 with respect to its geometrically finite subgroups (see [16], [17]). To this end we proceed as follows: the group H is geometrically finite as a result of Klein–Maskit free combination from H_1 and H_2, which are both geometrically finite subgroups of Γ_0. The $LERF$ property now says that for the element g_1 there exists a subgroup of Γ_0 of finite index which contains H and does not contain g_1. Call this subgroup Γ_1. Evidently, $g_1 \in [H_1, H_1] \subset \Gamma_1$ by construction. For the complete proof, see [14, Main Lemma]. □

Let us introduce the following notation: $\Omega_1^- = B_1^* \backslash \bigcup_{\gamma \in \Gamma_1} \gamma(\pi^-)$ where π^- is the component of $\mathbf{S}^3 \backslash \pi$ for which $w_3 \in \pi^-$. Let $\Gamma_1' = \mathrm{Stab}(\Omega_1^-, \Gamma_1)$.

The complete proof of the following assertion can be also found in [14, Lemma 3].

Lemma 3 *The group $G_1 = \langle \Gamma_1', \ \tau_\pi \Gamma_1' \tau_\pi \rangle$ is discontinuous and*

(1) $G_1 \cong \Gamma_1' *_{H_2} (\tau_\pi \Gamma_1' \tau_\pi)$.

(2) *G_1 is isomorphic to a subgroup $G_1^* \subset R$ of finite index.*

Sketch of proof (1) This follows from the fact that the plane π is strongly invariant under H_2 in Γ_1' by [14, Lemma 3.c], which means $H_2 \pi = \pi$ and $\gamma\pi \cap \pi = \emptyset$, $\gamma \in \Gamma_1' \backslash H_2$. One can now get assertion (1) from Maskit's First Combination theorem [11].

(2) Consider the reflection τ_{w_2} in the geodesic plane $w_2 \subset B_1$. We claim that the group $G_1^* = \langle \Gamma_1', \tau_{w_2} \Gamma_1' \tau_{w_2} \rangle$ is isomorphic to G_1. Indeed, w_2 is also strongly invariant under H_2 in Γ_1' and we again observe that $G_1^* = \Gamma_1' *_{H_2} (\tau_{w_2} \Gamma_1' \tau_{w_2}) \cong G_1$ because $\tau_{w_2} |_{w_2} = \tau_\pi |_\pi = id$.

Now $\tau_{w_2} \in R$. Therefore, $G_1^* \subset R$ and G_1^* has a compact fundamental domain $R(G_1^*) = R(\Gamma_1') \cap \tau_{w_2}(R(\Gamma_1'))$. The covering $\mathbf{H}^3 / (G_1^* \cap \Gamma_0) \to \mathbf{H}^3 / G_1^*$ is finite since $|R : \Gamma_0| < \infty$ and, hence, the manifold $M(G_1^* \cap \Gamma_0) = \mathbf{H}^3 / (G_1^* \cap \Gamma_0)$ is compact. Thus, the covering $M(G_1^* \cap \Gamma_0) \to M(\Gamma_0)$ is finite as well and so $|\Gamma_0 : G_1^* \cap \Gamma_0 | < \infty$. □

Corollary 4 *There exists a path α_t: $[0,1] \to Def(G_1^*)$ such that $\alpha_0 = G_1^*$ and $\alpha_1 = G_1$.*

Proof By choosing a continuous family of spheres μ_t for which $\mu_t \cap \pi = w_2 \cap \pi = \Lambda(H_2)$, $\mu_0 \supset w_2$, $\mu_1 = \pi$, $t \in [0,1)$, we construct the family of groups $G_t = \langle \Gamma'_1, \tau_{\mu_t}\Gamma'_1\tau_{\mu_t}\rangle$ by the arguments of Lemma 3. Consider now the action of Γ'_1 in B_1^* where $p_1\colon B_1^* \to B_1^*/\Gamma_1$ is the covering map. The surfaces $p_1(\mu_t)$ are all embedded and parallel due to condition (b). If now Ω_{G_t} is the component of G_1 containing ∞ then the manifold $M_{G_t} = \Omega_{G_t}/G_t$ is homeomorphic to the double of the manifold $M_1^- = \Omega_1^-/\Gamma'_1$ along the boundary $p_1(\pi)$. Thus, for all $t \in [0,1]$, M_{G_t} are all homeomorphic and there exists a continuous family of homeomorphisms $f_t\colon \Omega(G_1^*) \to \Omega(G_t)$ such that $G_t = f_t G_1^* f_t^{-1}$, $G_1 = f_1 G_1^* f_1^{-1}$. □

By construction the domain $R(G_1) = R(\Gamma'_1) \cap \tau_\pi\left(R(\Gamma'_1)\right)$ is fundamental for the action of G_1 in Ω_{G_1}.

Claim 5 $R(G_1) \cap \Sigma = \left(\mathcal{P}(H_3) \cup I_{g_1} \cup I'_{g_1}\right) \cap \Sigma$.

Proof Recall that $\pi^+(\pi^-)$ means the right (left) component of $\mathbf{S}^3\backslash\pi$ $(I_{g_1} \in \pi^+)$. Then $\pi^+ \cap \Sigma \cap R(\Gamma'_1) = \mathcal{P}(H_1) \cap \Sigma = \left(I_{g_1} \cup I'_{g_1}\right) \cap \Sigma$ by (b) and (c) of Lemma 2.

Also, $\tau_\pi\left(\pi^- \cap \Sigma \cap \tau_\pi(R(\Gamma'_1))\right) = \pi^+ \cap \tau_\pi(\Sigma) \cap R(\Gamma'_1) \subset \mathcal{P}(H_1) \cap \Sigma'$, so $\pi^- \cap \Sigma \cap R(G_1) = \tau_\pi\left(\mathcal{P}(H_1)\right) \cap \Sigma = \mathcal{P}(H_3) \cap \Sigma$. □

Let us consider now the family of spheres Σ_t centered on the y–axis (figure 2) such that $\Sigma_t \cap w_3 = \Sigma \cap w_3$, $\sigma_1 = \Sigma$, $\sigma_0 = \Sigma_0$, $t \in [0,1]$, where $\Sigma_t \cap \mathrm{ext}(B_1) \cap \mathrm{ext}(B_2) \subset \mathrm{ext}(\Sigma) \cap \mathrm{ext}(B_1) \cap \mathrm{ext}(B_2)$ (recall $\mathrm{ext}(\cdot)$ is the exterior of a set in $\overline{R}^3$), $\Sigma_t \cap I_{g_1} = \emptyset$ $(t > 0)$. Denote by τ_{Σ_t} the corresponding reflections. As before take the domain $\Omega^* = \Omega_{G_1}\backslash G_1(\Sigma_0^-)$ and the group $G'_1 = \mathrm{Stab}(\Omega^*, G_1)$, where $\Sigma_0^- = \mathrm{ext}\ (\Sigma_0)$ is the unbounded component of $\overline{R}^3\backslash\Sigma_0$.

Denote $\mathcal{G}_t = \langle G'_1,\ \tau_{\Sigma_t} G'_1 \tau_{\Sigma_t}\rangle$. Evidently, $\mathcal{G}_1 = \lim_{t\to 1} \mathcal{G}_t$.

Lemma 6 *The groups $\mathcal{G}_t$ are discontinuous, $t \in [0,1]$.*

Proof First, let us prove the lemma for $t \neq 1$. By Claim 5 we have now that $R(G_1) \cap \Sigma_t = \mathcal{P}(H_3) \cap \Sigma_t$. Moreover we claim also that

$$\begin{aligned} g\Sigma_t \cap \Sigma_t = \emptyset,\ g \in G_1\backslash H_3,\ H_3\Sigma_t = \Sigma_t, \\ \text{where} \quad H_3 = \tau_\pi H_1 \tau_\pi\ . \end{aligned} \tag{3}$$

To prove (3) we only need to show that $g(\Sigma_t \cap \Lambda(H_3)) \cap (\Sigma_t \cap \Lambda(H_3)) = \emptyset$, but this can be shown from the fact that each point of $\Lambda(H_3)$ is a point of approximation (see [14, Claim 1]).

All conditions of Maskit's First Combination theorem are now satisfied for the groups G'_1 and $\tau_{\Sigma_t} G'_1 \tau_{\Sigma_t}$ $(t \neq 1)$ [11] and we obtain also

$$\mathcal{G}_t \cong G'_1 *_{H_3} (\tau_{\Sigma_t} G'_1 \tau_{\Sigma_t}) \tag{4}$$

where the $\mathcal{G}_t$ are all discontinuous, $t \in [0,1)$.

Let us now consider the group $\mathcal{G}_1$ and the domain $R(\mathcal{G}_1) = R(G_1) \cap \tau_\Sigma(R(G_1))$. Our goal now is to show that $R(\mathcal{G}_1)$ is a fundamental domain for the action of $\mathcal{G}_1$ in $\Omega_{\mathcal{G}_1}$ $(\infty \in \Omega_{\mathcal{G}_1})$. If now $\langle g_1, \gamma_1, \ldots, \gamma_\ell \rangle$ is a set of generators of G'_1 then $S = \langle g_1, \gamma_1, \ldots, \gamma_\ell,\ g_2, \gamma'_1, \ldots, \gamma'_\ell \rangle$ are generators of $\mathcal{G}_1$, where $\gamma'_i = \tau_\Sigma \cdot \gamma_i \cdot \tau_\Sigma$ and $g_2 = \tau_\Sigma \cdot g_1 \cdot \tau_\Sigma$. Observe that the element g_1 is included in S because some of its isometric spheres belong to the boundary $\partial R(G'_1)$

We want to apply the Poincaré Polyhedron theorem [12]. Indeed, an arbitrary cycle of edges in $\partial R(\mathcal{G}_1)$ consists either of edges situated in $\partial(R(G_1)) \cap \mathrm{int}(\Sigma)$, and $\partial(\tau_\Sigma(R(G_1))) \cap \mathrm{ext}(\Sigma)$, or is an edge cycle $\ell_1 = I_{g_1} \cap I_{g_2}$, $\ell_2 = I'_{g_1} \cap I'_{g_2}$, where I_{g_k}, I'_{g_k} are the isometric spheres of g_k and $g_k^{-1} (k = 1,2)$. The sum of angles in any cycle of the first type is 2π because $R(G_1)$ is a fundamental domain [12].

We now claim that the element $g = g_2^{-1} \cdot g_1$ is parabolic with a fixed point $d = I_{g_1} \cap I_{g_2}$. Indeed, $g_2^{-1} \cdot g_1 = \left(\tau_\Sigma \cdot \tau_{I_{g_1}}\right)^2$ because $g_1 = \tau_{\pi_2} \cdot \tau_{I_{g_1}}$ and π_2 is orthogonal to Σ (figure 2). Now it is easy to check that $g(d) = d$, $gI_{g_1} \subset \mathrm{int}(I_{g_2})$ and $g(\mathrm{int}(I_{g_1})) = \mathrm{ext}\ (g(I_{g_1}))$, therefore the elements g and $g' = g_1 \cdot g \cdot g_1^{-1}$ are parabolics.

All conditions of the Maskit–Poincaré theorem are valid at the edges ℓ_i also and, hence, $\mathcal{G}_1$ is discontinuous. Lemma 6 is proved. □

Lemma 7 *The group $\mathcal{G}_0$ is isomorphic to a subgroup $L' \subset R$ of a finite index.*

Proof We repeat our construction of $\mathcal{G}_0$ by modelling it in $\mathbf{H}^3$ so as to get the required isomorphism.

Recall that we started from the group $\Gamma'_1 \subset \mathrm{Isom}(\mathbf{H}^3)$ and showed that $G_1 = \langle \Gamma'_1, \tau_\pi \Gamma'_1 \tau_\pi \rangle \cong G_1^* = \langle \Gamma'_1, \tau_{w_2} \Gamma'_1 \tau_{w_2} \rangle$ (see Lemma 4). Next we constructed $\mathcal{G}_0$ by using reflection in $\sigma_0 = \Sigma_0$ such that $\sigma_0 \cap w_3 = \Lambda(H_3)$, $\sigma_0 \cap B_1 = \emptyset$, $w_3 = \tau_\pi(w_1)$.

Let $\eta = \tau_{w_2}(w_1) \subset \mathbf{H}^3$, $\eta \in W$. Again let us take the subgroup G_1^{**} of G_1^* which is $G_1^{**} = \mathrm{Stab}(\mathbf{H}^3 \backslash G_1^*(\eta^-),\ G_1^*)$, where η^- is a subspace $\mathbf{H}^3 \backslash \eta$ not containing w_2.

By construction the fundamental domain $R(G_1^*) = R(\Gamma_1') \cap \tau_{w_2}(R(\Gamma_1'))$ of the group G_1^* satisfies $R(G_1^*) \cap \eta = \mathcal{P}(H_3' = \mathrm{Stab}(\eta, G_1^*))$. Again by Maskit's First Combination theorem we have a group L':

$$L' = G_1^{**} *_{H_3'} (\tau_\eta G_1^{**} \tau_\eta) \tag{5}$$

We constructed an isomorphism $\varphi_1\colon G_1^* \to G_1$ in Lemma 4 such that $\tau_\pi \cdot \varphi_1 \cdot \tau_{w_2} = \varphi_1$, therefore $\varphi_1(H_3') = H_3$ and $\varphi_1(G_1^{**}) = G_1'$. It follows now from (4) and (5) that the map $\varphi_1\big|_{G_1^{**}}$ can be extended to an isomorphism $\varphi\colon L' \to \mathcal{G}_0$.

Index $|R : L'|$ is finite because L' has a compact fundamental domain. The Lemma is proved. □

Recall that we identify $[\rho] \in \mathrm{Def}(L')$ with $\rho(L')$.

Lemma 8 *There exists a path $\beta_t\colon [0,1] \to cl(\mathrm{Def}(L'))$ such that $\beta_0 = L'$, $\beta_1 = \mathcal{G}_1 \in \partial\,\mathrm{Def}(L')$, $\beta_t([0,1)) \subset \mathrm{Def}(L')$.*

Proof We have constructed a path $\alpha_t\colon [0,1] \to \mathrm{Def}(G_1^*)$ in Corollary 4 such that $\alpha_0 = G_1^*$, $\alpha_1 = G_1$ and α_t is a family of admissible representations. Let further $\alpha_t\big|_{G_1^{**}} = \alpha_t'$. Obviously, the representations α_t' are also admissible and $\alpha_1'(G_1^{**}) = G_1'$. We can easily extend our family α_t' to a family of admissible representations $\theta_t\colon L' \to \mathrm{Def}(L')$ by the formula $\theta_t = \tau_{\mu_t}\alpha_t'\tau_{\mu_t}$, where μ_t are the spheres constructed in Corollary 4.

Observe that $\mu_1 = \pi$ and now take a new continuous family of spheres ν_t for which $\nu_t \cap w_3 = \Lambda(H_s) = w_3 \cap B_2$ and $\nu_1 = \tilde{w}_3$, $\nu_2 = \Sigma_0$ where $\tilde{w}_3$ is the sphere containing w_3 $(t \in [0,1])$.

Again we have a path $\theta_t'(L') = \langle G_1', \tau_{\nu_t} G_1' \tau_{\nu_t}\rangle$. Composing the path θ_t with θ_t' and with the path corresponding to spheres Σ_t connecting Σ_0 with Σ_1 we get required path β_t. The Lemma is proved. □

4 Proof of Theorem 1

(1) Denote by $F = \pi_1\sigma$ a fixed fiber group of our initial manifold M, and let also $F_0 = \Gamma_0 \cap F$.

By Jørgensen's theorem [5] the limit $\beta_1 = \lim\limits_{t\to 1} \beta_t$ is an isomorphism $\beta_1\colon L' \to \mathcal{G}_1$. Let us consider the subgroup $L = L' \cap \Gamma_0$, $|\Gamma_0 : L| < \infty$. Put also $F_L = L \cap F_0$ for its normal subgroup. We have also the curve $\beta_t(L) \subset \mathrm{Def}(L)$. Let $\mathcal{N} = \beta_1(L)$, $\mathcal{F} = \beta_1(F_L)$. Let us show that $g = g_2^{-1} \cdot g_1 \in \mathcal{F}$. To this

end let us recall that the element g_1 was chosen from the very beginning being in $[H_{w_1}, H_{w_1}]$ (Lemma 1). Recalling also that $\beta_1^{-1}(g_1) = g_1$ and denoting $\beta_1^{-1}(g_2) = g_2'$, by construction we get $g_2' = \tau_\eta \cdot g_1 \cdot \tau_\eta$, $\eta = \tau_{w_2}(w_1)$, $g_1 \in [H_{w_1}, H_{w_1}] \subset [F_0, F_0]$ (see Lemma 1). The group Γ_0 was chosen to be normal in the reflection group R, and since $[\Gamma_0, \Gamma_0] \subset F$, it is straightforward to see that

$$r[F_0, F_0]r^{-1} \subset F_0, \quad r \in R \ .$$

Hence, $g_2' \in F_0$, and for the element $g' = (g_2')^{-1} \cdot g_1$ we immediately obtain $g' \in F_L = F_0 \cap L'$. It follows that $\beta_1(g') = g = g_2^{-1} \cdot g_1 \in F_0 \cap \mathcal{G}_1 = \mathcal{F}$ as was promised.

We have that $\mathcal{N}$ is isomorphic to the semi-direct product of $\mathcal{F}$ and the infinite cyclic group $\mathbf{Z}$, so taking the element $t \in \mathcal{N} \backslash \mathcal{F}$ projecting to the generator of $\mathcal{N}/\mathcal{F}$, we observe that the elements

$$g_n = t^n g t^{-n} \in \mathcal{F} \ , \quad g \in \mathcal{F}, \quad n \in \mathbf{Z} \tag{6}$$

are all parabolics. Since $\mathcal{N}$ contains no abelian subgroups of rank bigger than 1 and $t^n \notin \mathcal{F}$ $(n \in \mathbf{Z})$ one can easily see that the elements (6) are also non-conjugate in $\mathcal{F}$. We have proved (1) of the Theorem.

(2) By the construction, the fundamental polyhedron $R(\mathcal{G}_1)$ of the group $\mathcal{G}_1$ contains only one conjugacy class of parabolic elements g of rank 1. There is a strongly invariant cusp neighborhood $B_g \cong [0,1] \times R^1 \times [0, \infty)$ which comes from the construction of $R(\mathcal{G}_1)$. So each parabolic g_n of type (6) gives rise to submanifold

$$B_{g_n} / \langle g_n \rangle \cong T_n \times [0, \infty), \ T_n \cong S^1 \times S^1 \tag{7}$$

in the manifold $M(\mathcal{F}) = \Omega_N / \mathcal{F}$. Therefore $M(\mathcal{F})$ contains infinitely many parabolic ends (7) bounded by tori T_n. They all are non-parallel in $M(\mathcal{F})$ and therefore by Scott's "core" theorem the group $\pi_1(M(\mathcal{F}))$ is not finitely generated [16]. $\square$

Remark By using the argument of [14] one can prove:

Theorem 2 *There is a (non-faithful) represention $\beta_{1+\varepsilon}$ which is ε–close to β_1 for some small $\varepsilon > 0$ such that the group $\beta_{1+\varepsilon}(F_L)$ is infinitely generated, has infinitely many non-conjugate elliptic elements. Moreover, $\beta_{1+\varepsilon}(F_L)$ is a normal infinitely presented subgroup of a geometrically finite group $\beta_{1+\varepsilon}(L)$ without parabolics.*

To prove the theorem one can continue to deform the group for $1 < t \leq 1+\varepsilon$ (these representations will no longer be faithful) in order to get an elliptic element g_t whose isometric spheres form an angle $\theta(t)$ instead of being tangent. To do this in our Lemma 2, instead of the sphere Σ tangent to the isometric spheres of g_1, one needs to consider a nearby sphere $\Sigma_{1+\varepsilon}$ forming angle $\theta(\varepsilon)$ with them. If $\theta(\varepsilon) = \frac{\pi}{2n}$ and $n > 0$ is large enough the group $\beta_{1+\varepsilon}(F_L)$ is Kleinian, has infinitely many non-conjugate elliptic elements of the order n (obtained as above as an orbit of $g_{1+\varepsilon}$ by a pseudo-Anosov automorphism of the $\beta_{1+\varepsilon}(F_L)$). The construction gives us that $\beta_{1+\varepsilon}(F_L)$ is a normal and finitely generated but infinitely presented subgroup of the geometrically finite group $\beta_{1+\varepsilon}(L)$ without parabolic elements. In particular $\beta_{1+\varepsilon}(L)$ is a Gromov hyperbolic group (see [14, Lemmas 5–7]).

References

[1] **L V Ahlfors**, *Finitely generated Kleinian groups*, Amer. J. Math. 86 (1964) 413–429; 87 (1965) 759

[2] **L Bers**, *On boundaries of Teichmüller spaces and on Kleinian groups I*, Annals of Math. 91 (1970) 570–600

[3] **B Bowditch**, *Geometrical finiteness of hyperbolic groups*, preprint, University of Melbourne

[4] **B Bowditch**, **G Mess**, *A 4–Dimensional Kleinian Group*, Trans. Amer. Math. Soc. 344 (1994) 391–405

[5] **T Jørgensen**, *Compact 3–manifolds of constant negative curvature fibering over the circle*, Annals of Math. 106 (1977) 61–72

[6] **M Kapovich**, *On Absence of Sullivan's cusp finiteness theorem in higher dimensions*, preprint

[7] **M Kapovich**, **L Potyagailo**, *On absence of Ahlfors' finiteness theorem for Kleinian groups in dimension* 3, Topology and its Applications, 40 (1991) 83–91

[8] **M Kapovich**, **L Potyagailo**, *On absence of Ahlfors' and Sullivan's finiteness theorems for Kleinian groups in higher dimensions*, Siberian Math. Journal 32 (1992) 61–73

[9] **D Long**, *Immersions and embeddings of totally geodesic surfaces*, Bull. London Math. Soc. 19 (1987) 481–484

[10] **B Maskit**, *On boundaries of Teichmüller spaces and on Kleinian groups, II*, Annals of Math. 91 (1970) 608–638

[11] **B Maskit**, *On Klein's Combination theorem III*, from: "Advances in the theory of Riemann Surfaces", Princeton Univ. Press (1971) 297–310

[12] **B Maskit**, *Kleinian groups*, Springer–Verlag (1988)

[13] **J Morgan**, *Group action on trees and the compactification of the space of conjugacy classes of* $SO(n,1)$*–representations*, Topology 25 (1986) 1–33

[14] **L Potyagailo**, *Finitely generated Kleinian groups in 3–space and 3–manifolds of infinite homotopy type*, Trans. Amer. Math. Soc. 344 (1994) 57–77

[15] **L Potyagailo**, *The problem of finiteness for Kleinian groups in* 3*–space*, from: "Proceedings of International Conference, Knots-90", Osaka (1992)

[16] **P Scott**, *Finitely generated* 3*–manifold groups are finitely presented*, J. London Math. Soc. 6 (1973) 437–440

[17] **P Scott**, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. 17 (1978) 555–565; Correction ibid 32 (1985) 217–220

[18] **D Sullivan**, *Travaux de Thurston sur les groupes quasi-fuchsiens et les varietes hyperboliques de dimension* 3 *fibres sur* S^1, Lecture Notes in Math, 842, Springer–Verlag, Berlin–New York (1981) 196–214

[19] **W Thurston**, *The geometry and topology of* 3*–manifolds*, Princeton University Lecture Notes (1978)

Département de Mathématiques
Université de Lille 1
59655 Villeneuve d'Ascq, France

Email: `potyag@gat.univ-lille1.fr`

Received: 20 November 1997 Revised: 7 November 1998